# AN ESSAY

ON THE

# THEORY AND PRACTICE

OF

# SETTING OUT

# Railway Curves.


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# PREFACE.

THE Manuscript Notes from which, with some added explanatory matter, this Pamphlet is printed, were made for private use; and are published with a view to their being possibly both useful and interesting to others.

The plan due to Mr. RANKINE is almost exclusively preferred and elucidated. Two others are however given and fully explained; together with directions regarding S Curves and intermediate portions of straight in effecting a serpentine Junction.

# ELEMENTARY INTRODUCTION.

THERE are two formulæ given in the directions and traced to their rationale; these formulæ, the more informed reader will excuse our premising, depend upon certain geometrical relations subsisting between lines and angles in and about a circle, and are common to all circles.

The first formula serves to calculate the radius of the curve (say AO) *from* the sine AH, and from the measure of its angle AOH, (the angle to which AH is the ascertained *actual* sine;) the other formula is in like manner the summary of all the arithmetic necessary to find the angle $\beta$ (or rather its equal the half of the angle at the centre of the circle in the same segment, viz. AO 1) *from* the radius of the curve, and half the subchord A 1. (See diagram p. 8.)

What is meant by the tabular sine of so many degrees, is, the length given in a "table of sines," as the sine of an angle of that dimension or angular quantity. These tables are calculated according to the proportion which the length of the radius of a circle, or sine of 90°, bears to that of the sines of all other angles at the centre of the same circle. The sine of 90° as the longest sine possible, is fixed at a certain length, which in our tables of "Natural Sines" is thus expressed, 1.00000000, which may be taken to represent unity, and then the sine of any other angle must be less than unity, or a decimal fraction. In logarithmic tables of sines, radius is taken to be 10., which is the logarithm of 10,000,000,000, and the other sines are represented by logarithms of numbers in proportion to that great length of radius; the same remarks

apply to co-sines and tangents. It must be clear then that if we have the sine of a certain angle, say 15.35, in the field, and the angle itself, say 12°45, the radius of the circle is attainable by the rule of three thus, by logarithms:

| | | | | |
|---|---|---|---|---|
| As | log. | sine 12°45' | = | 9.3437973 |
| : | — | 15.35 chains | = | 1.1161084 |
| : : | — | sine 90° | = | 10. |
| : | — | 69.553 chains | = | 1.8423111 |

Which might be done by natural arithmetic, thus:

| | Natural sine 12°45' | | Chains | | Rad. |
|---|---|---|---|---|---|
| As | .2206974 | : | 15.35 | : : | 1. |

$$\begin{array}{r} 1 \\ \hline .2206974\,)\ 15.3500000\ (69.553 \\ 13241844 \\ \hline 21081560\ \ \&c. \end{array}$$

We have descended to the very elementary principles connected with the matter, but such instruction will not appear contemptible to those for whose benefit it is intended. It is indeed too common with the elder members of a profession to lose sight of elementary principles, and many young men enter the office of a surveyor or engineer ill prepared by mathematical acquirements, to whom the study of this little pamphlet may prove of ulterior advantage and interesting, by showing that there is a great deal of mathematics in even a simple operation, and perhaps suggesting enquiry into elementary principles of our practice generally as the best means of sharpening invention.

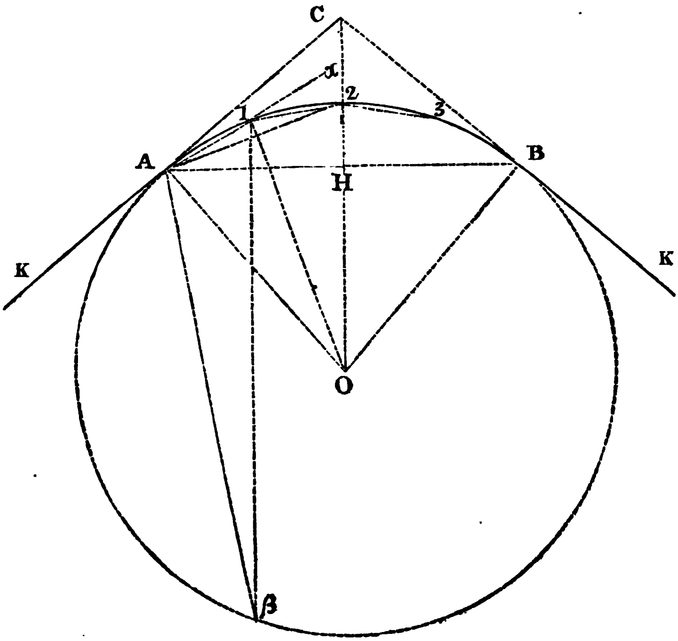
# RAILWAY CURVES.

## THE SINGLE CURVE.

If two straight portions of a line are required to be united within the angle of their intersection, their junction is most equably and practically effected by a curve — which is an arc of their common tangent circle. To find which,

Let KA and KB be two straight portions of the central line of a proposed Railway. It is required to set out points of the curve, at ten feet or a hundred feet, or a chain, or any specified distance asunder, such curve being an arc of the only circle to which KA and KB, at their points of termination A and B, can be tangents.

The data then are the two lines KA and KB on the ground, and according to the plan we are adopting, our first objects are to determine their direction and thence the angle O at the centre of the circle, the distance AB, or chord of the arc, and generally thence to *calculate* the radius of the curve by formula No. 1: and then, by means of radius and the distance apart intended for the plugs that are to mark out the curve, to find by calculation, by means of another formula, the angular




quantity of $\beta$ (see the diagram) which will be proved to be equal to, or the double of, or supplemental to, several others: to be one or all used in the last operation—the driving of the plugs.

## DIRECTIONS IN SHORT.

AC must = BC
Find angle O by some means
Then radius of curve
And angle $\beta$

Find the point of intersection C if accessible; and if practicable, measure the angle ACB. The sup-

plement of the angle ACB = the angle AOB. Get the distance AB.

Find the radius of the curve by the following formula: call the radius $r$.

The tabular radius $R$

And the distance apart at which the points of the curve are to be plugged out on the ground. $\Big\} d$

**I.** then $r = \dfrac{AB \,.\, R}{2 \text{ sine } \frac{1}{2} \text{ angle } O}$*

The rationale of this formula is as follows:

$$\underset{\text{(of the Tables)}}{\text{As } R} : \underset{\text{(on the Ground)}}{r} :: \underset{\text{(of the Tables)}}{\text{sine } \tfrac{1}{2} \text{ angle } O} : \underset{\text{(the actual sine half angle O on the Ground)}}{\frac{AB}{2}}$$

Here are four terms in geometrical proportion; the product therefore of the two extremes equals the product of the two mean terms, thus:

$$r \text{ sine } \tfrac{1}{2} \text{ angle } O = \frac{AB \,.\, R}{2}$$

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* Which (for the sake of those who cannot read formulæ) in words means, that you are to *divide* the *product* of AB and the tabular radius (or sine of 90°) which in the table of *natural* sines is 1, by twice the tabular sine of half the angle O.

But if you work by logarithms, you have to *subtract* twice the logarithmic sine of half the angle O from the *sum* of the logarithm of AB, and the logarithmic tabular radius or sine of 90°

Dividing both terms of this equation by sine $\frac{1}{2}$ angle O, we have

$$r = \frac{AB \cdot R}{2 \text{ sine } \frac{1}{2} \text{ angle } O} \qquad \text{(as given)}$$

But supposing obstructions in the way of measuring the angle C, and the distances AC and BC, and that the ground between A and B *is* accessible.

Then measure AB, and get the angles CBA, CAB; (see that they be equal, if not, shift A or B to make them so;) their sum, being the supplement of C, equals the angle O.*

Again, supposing insurmountable difficulties in the way of measuring both the chord AB, and the angle CBA, or CAB, as well as C,

Then our first step will be to ascertain that A and B are really equi-distant points from the point of intersection C†, that is to say, whether they are the *correct tangent points,* because when that is adjusted, the centre O, if the ground is clear, may be determined by means of the intersection of two perpendiculars to the tangents, from A and B, and the radius can be measured, which with either angle will be sufficient to find A B.

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* Euclid 32 of 1;—corollary, "the four angles of a quadrilateral figure are equal to four right angles." In ACBO, the angles A and B are *two* right angles, the other two are divided therefore between C and O. One of these angles therefore is the supplement of the other.

† *It is in all cases first necessary to ascertain that the points A and B are equi-distant from the point of intersection C;* if not, that matter is easily adjusted by shifting the point A or B forwards in line towards C, or backwards (Euclid 32 of 3, whence corollary "straight lines drawn to touch a circle from a point without it are equal"). It is clear also that there is but one arc that, sweeping from A, and tangent to the straight portion of the line from which it springs, *can* be tangent to the other straight portion terminating at B; for any but *the* radius would either sweep short of B, or *cut* the line at that point, or go beyond it.

Our object at first being, it must be remembered, to find the *chord* A B, then *the radius*, from which to calculate a certain angle equal to that marked $\beta$ in the diagram.

Again supposing the centre inaccessible, and that the points A and B are not visible the one from the other:

1st. Take any two points, other than A and B, (in the tangents, or the tangents produced,) both points either above or below A and B, then the line of sight which joins them will make angles at the base of a triangle, of which the inaccessible point C is the apex. Half the sum of the internal angles being equal to CAB or BAC, and their sum being the supplement of C, is equal to the angle at O, the centre.

2nd. Lay off the angular quantity CAB from either of the above assumed points, and fix a point thereby on the opposite tangent, and measure the distance across, (call it $p\ q$) see that A and B are equi-distant from $p$ and $q$ respectively, and note the distance.

Then as sine C : $p\ q$ : : sine $p$ : C $p$ or C $q$

C p plus or minus A $p$ = AC or BC

Then as sine CAB : CB : : sine C : AB

With which, and the angle O, find by formula 1, the *radius*.

Find the angle $\beta$ (see diagram) by the following formula :—

$$\textbf{II.}\ \text{Angle}\ \beta = \frac{d\ .\ R}{2r}\ *$$

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* That is, divide the product of the *distance* the plugs are to be apart and the *tabular* radius or sine of 90° by the diameter of the *actual* circle. Or by logarithms: *subtract* the logarithm of the diameter from the *sum* of the logarithm of the *distance* ($d$) and logarithmic *tabular* radius.

The rationale of this formula is as follows,

As the actual radius . tabular radius . . actual sine $\frac{1}{2}$ angle AO1, . tabular sine $\frac{1}{2}$ angle AO1

(which is half subchord $d$)

Again state it thus:

$$\text{As } r \ : \ \text{Radius} \ :: \ \frac{d}{2} \ : \ \text{angle } \beta$$

$$\text{Whence angle } \beta = \frac{\text{Radius} \times \frac{d}{2}}{r} = \frac{d \cdot R}{2r}$$

(as given.)

And since the angle $\beta$ at the circumference is equal to $\frac{1}{2}$ angle AO 1, at the centre,

The degrees and minutes answering to this tabular sine $\frac{1}{2}$ angle AO 1, is therefore the measure of the angle $\beta$.

If the prescribed subchord ($d$) gives the angle B an awkward quantity to read off, it is sometimes altered, thus if $d$ at $\frac{1}{2}$ a chain, give $\beta = 56'\ 15''$, then

$$\text{As } \frac{56'\ 15''}{15} \ : \ \frac{50 \text{ links}}{16} \ :: \ \frac{60' = 1^\circ}{16}$$

$$15)\ 800$$

$$53\,.\,33$$

Thus $d$ may be made $53\frac{1}{3}$ links, and $\beta$ being $1^\circ$, can be laid off with greater ease and precision; it should however be observed, that this is not *mathematically* correct:—a slight error

is due to the *minute* difference between a very small increment of the chord, and the same proportionate very small increment of the arc, of no practical value.

This angular quantity $\beta$ equals *first,* the angle of deflection of the first subchord A 1 from the tangent CA 1, because it is an angle in the alternate segment (Euclid 32 of 3;) next, it is equal to 12 A in the same segment, and then to 1 A 2, 2 A 3, &c., in similar segments of the same circle. The angle $x$ 1 2 is the double of it, &c. See General Observations.

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## THE S CURVE.

If the junction of two straight portions of a line must be effected outside the angle of their intersection C, the line of junction has two turns, and necessarily takes a serpentine course formed by *two* arcs tangent to each other and to the opposite sides of the two straight portions of the Railway; and if the latter are in direction parallel to each other, the two most eligible curves will be similar arcs of equal circles, such therefore whose common tangent bisects a straight line joining the ends of the two straight portions of the Line; or otherwise two unequal arcs of equal circles.

If not parallel, and if both the radius of one arc and point of mutual contact be given, the other arc must be made subordinate to it, because the

point of its contact with the straight portion of the Railway must be "adjusted" subordinately to the direction of the common tangent.—(See Note † page 10.)

And if the *two tangent points* and *one radius* are predetermined, then erect the given length of radius perpendicular to *both* tangents at their points of contact with the two straight portions of the line, and on the *same side* of the Railway course, and join their terminations by a straight line, which bisect; and then a perpendicular from the point of bisection will intersect the given radius produced at the centre of the second circle, and thus the elements of both circles are attained.

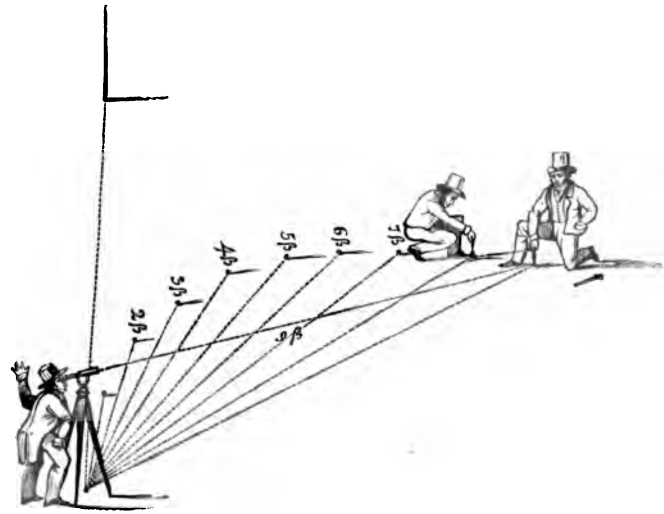
If the two tangents only are given, assume a radius, or assume a point of contact, or both, and proceed as in the two last cases given.

Proceed in the setting out of these two several curves, in all respects, as with one.

INTERMEDIATE STRAIGHT.—If any considerable inequality in the sharpness of the two arcs is unavoidable, either from the nature of the country or direction of the tangents, then the safety of the line, for the purposes of rapid transit, generally requires the continuation of the course of the Railway by means of the intermediate tangent for some distance, say 10 chains, or less—to which, and the further tangent, the second curve is then adjusted.

# FIELD DIRECTIONS.

Set up your Theodolite at A, then set the arrow of the upper plate at zero on the graduated limb of the lower plate, and lay off towards B the quantity $\beta$ for the direction of the first subchord;—the distance $d$ measured in such direction, will fix point the first; drive here your plug 1, which examine and adjust by the instrument, and let your men be particular




as to the distance; now lay off $2\,\beta$ from AC, and let the distance $d$ be extended from plug 1, so as to meet the line of collimation of the instrument,

your hand directing the chainman's hand to the point, and here drive and adjust plug 2, and so on to the end of the curve. Or,

If the ground or the length of the curve be unsuitable for laying off all the angles at A, as above directed; a very little ingenuity and trigonometry will give other means of finding the points sought.

It is clear from what has been already explained, that the *angle of deflection* $x$ 1 2 (that is the angle made by the line of direction of one subchord with that of the next subchord) $= 2\,\beta$, for it is equal to the two internal and opposite angles in the triangle A 1 2. (Euclid 32 of 1.)

And the adjacent angle, (as A 1 2) to this angle of deflection, viz.: the internal angle at the intersection of the subchords is its supplement; that is

$$x\ 1\ 2 = 180^\circ - A\ 1\ 2;\ \text{or}\ 180^\circ - x\ 12 = A\ 1\ 2.$$

Thus, Mr. Rankine's plan, in practice, is simple enough.

After having calculated the angle $\beta$, you have to radiate from AC at the point A, the successive angular quantities $\beta$, $2\,\beta$, $3\,\beta$, &c., to the end of the curve, and direct the men first to measure $d$, in the line of collimation at the first reading $\beta$; and ever after to extend the measure $d$, from plug to point of intersection with your successive radiating lines of collimation, to which precisely you are to direct the hand of your leading chainman.

This field work may be stated thus: in the formation of a succession of triangles, standing upon contiguous subchords of our curve, we have a side fixed on the ground, also the contiguous angles subtending the subchords, and thence the *direction* of a second and *longest* side of each triangle, (Euclid 15 of 3,) then from the point of our curve last fixed we extend the *length* of the third side of our triangle, viz. the subchord *d* as a radius, and at the *ulterior* point of its incidence, upon the line of collimation, we fix the third angle of the triangle and a new and true point of the curve.

You may here and there, if necessary from the nature of the ground, move on your instrument to the last plug driven in, and take the angle of deflection for the next chord, and then go on radiating the quantity $\beta$ again, as far as convenient, and then perhaps move the instrument to the penultimate plug, and proceed as at A, radiating $\beta$, 2 $\beta$, &c., or if necessary take the internal angle at the intersection of the subchords, viz. $180^\circ - 2\,\beta$.

One or other of the modes of applying this simple and excellent mode of setting out a curve, will be found, it is believed, to snit any ground or any curve. We have endeavoured to be thoroughly explicit on this one plan, which we think the best; every other being more complicated, or not adapted to the sort of country in which curves are generally required.

Its accuracy depends on lines of sight (independent of broken or uneven surface,) in it we avail ourselves of the powerful aid of the Theodolite; (an observation however that applies in a still greater degree to the plan sometimes adapted

with two Theodolites, which is described under the head of General Observations; where also is explained a plan which has the merit of simplicity, though it is not often available.) An instrument of the ordinary portable size, of five or six inch plate, well divided on silver, (with, say a check arrow opposite the vernier against *great* errors,) will lay off angles certainly to the minute, and greater precision is soon acquired; but granting a minute of *error*, on either side, what does it amount to, in comparison with the probable errors in setting out a curve by the successive productions of subchords to a *calculated extent* with calculated offsets?

On (say) a chain, or one hundred feet, what is a minute error? or in other words, what is the extent of an arc of 1' of a circle of two chains, or of one two hundred feet in diameter?

$$\text{Circum,} = \frac{2 \times 66 \times \overset{\text{Inches.}}{12} \times 3.1416}{360^\circ \times 60} = .23084 = \text{less than } \tfrac{1}{4} \text{ in. at 1 chain off.}$$

$$\frac{3.1416 \times \overset{1}{\cancel{24.00}}}{\underset{9}{\cancel{216.00}}} = .34906 = \text{nearly } \tfrac{7}{20} \text{ in. on 100 feet radius.}$$

Precision even to this extent is wonderful, and due to our mathematical artizans, Peter Vernier and the Gossamer Spider.

# GENERAL OBSERVATIONS.

By inspecting the diagram and recollecting Euclid 8 of 6, the student will discover, first, 6 similar triangles, making three pairs of equal triangles, viz. CAO, and CBO, and the four smaller that compose them, viz. CHB, CHA, HAO, HBO; any angle in one of which is equal to an angle in each of the others; several sides are *common* to two, though not homologous sides, &c.

2nd. The angle C is the supplement, and the angle O is equal to the sum of any two angles, at the base of a triangle, formed by any straight line, extending from any point in line with AC, and any other point in line with BC; and their sum is equal to the sum of CAB and CBA; and of the *external* angles half the sum = KAH or KBH.

3rd. There are interminable relations subsisting between the lines and angles in this diagram, viz.: (Euclid 8 of 6)

AC and CB are each a mean proportional between CO and CH
AO and OB - - - - - - - CO and OH
AH and HB - - - - - - - CH and HO
Also, AC = CB. AO = OB, &c.

So that if we can *measure* a side and angle, or two sides of any of those six triangles, all the other lines and angles in the whole diagram can be ascertained.

Also any part of the angle, formed by tangent and chord, being laid off from the tangent AC, the line of collimation will cut off the same proportional part of the arc, and will be

intersected at the circumference by the line of collimation of an instrument laying off a quantity equal to the *remainder* of that angle from the other tangent BC, or to such *part* of it from the chord BA; thus, if $\frac{1}{3}$ angle CAB be laid off from AC, and $\frac{2}{3}$ from BC, the lines of collimation will intersect at the curve, and be respectively chords of $\frac{1}{3}$ and $\frac{2}{3}$ the curve.

This leads us to the plan by two Theodolites, set up at the two ends of the curve, which radiating simultaneously from chord of the whole arc AB and tangent respectively, the angular quantity $\beta$, 2 $\beta$, 3 $\beta$, &c., fix at the intersection of their lines of collimation all the points of the curve at the distance *(d)* independently of measurement: this plan is adapted to sharp turns over awkward ground.

The geometry of the radius, subchord (*d*), and the angle $\beta$, being the same as in the plan explained in the text. In one case, each plug is driven at the intersection of two sides of a triangle by means of the length of one, and the direction in which lies the termination of the other; by the other plan, the *identical* points are found by the intersection of a chord of a portion of the arc and the chord of the remainder, the *direction only* of both being given; such chords and the chord of the whole arc which is fixed on the ground, forming a triangle.

That the intersection of two subchords (unequal unless they meet at the centre of the curve) so radiating, *pari passu*, from tangent* and chord of the whole arc respectively, *are* points in an arc of a circle, can be shewn thus—these subchords, at every remove, *increase* one angle at the base of the triangle they form with the chord of the whole arc, and

decrease the other equally (from the way in which they are laid off) so that their *sum* is constantly the same, and, consequently, their supplement, the quantity of the angle at the moving apex of the varying triangle is *constant*.

All such equal angles, being upon the chord, are therefore in the *segment* of a circle: (deduced from Euclid 21 of 3.) So that the motion of the apex may be said to generate the arc; and it is a segment of the *right circle*, because every two straight lines which form an angle in it are *set off*, each at an angle from the tangents respectively, equal to the angle in the alternate segment; thus, CA1 $= \beta$, or if $\frac{1}{3}$ CAB were set off from AC and $\frac{2}{3}$ from BC (which is the same in effect as $\frac{1}{3}$ from AB) the angles in the alternate segments would be respectively $\frac{1}{3}$ and $\frac{2}{3}$ CAB.

A few words will now explain a plan which has the claim of great simplicity, and suits a wide curve from every point of which the centre is visible. Certain points, (call them 1, 2, 3, &c., not in our diagram,) at any equal distances apart, are set off on the *tangent* from A and from B towards C, whence offsets are projected, all towards the centre; the length of these offsets is all we want—put O for the centre.

Then OA1, OA2, OA3, &c., are right angled triangles. (Euclid 16 of 3.)

And by 47 of 1st

$\sqrt{rad^2 + A\,1^2} = O\,1$, and $O\,1 - r =$ 1st offset

$\sqrt{r^2 + A\,2^2} - r =$ 2nd offset.

$\sqrt{r^2 + A\,3^2} - r =$ 3rd offset, &c.

If the offsets are inconveniently long, bisect or trisect the curve upon the principle suggested above: at the points of bi or tri-section, erect sub-tangents perpendicular to the radii from these points, and mark their points of intersection with the first tangents AB, AC, and each other. On each pair of sub-tangents so obtained, set off distances and calculate offsets as before: these being the same in all. The offsets give points in the curve, though not equi-distant.

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The student may trace many other relations which subsist between the lines and angles about a curve; such as, that what is called the angle of deflection of a chord of any part of an arc, from the chord produced of the remainder, is always equal to that of the chord of the whole arc from the tangent at either extremity of the chord of the whole arc; for being the external angle of the produced side of a triangle, it is equal to the two interior and opposite angles, which together equal the angle made by chord and either tangent.

Thus* $x\,1\,B = 1\,AB + 1\,BA = CAB$ or $CBA$.

In curving, as in all other practice, the Engineer has

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* Imagine a straight line joining 1 B.

constant need for invention and little contrivances, as well as for an exact knowledge of the mathematical principles of what he is about; for instance, to obviate obstructions such as garden walls, &c., he may provide a piece of prepared tape, of the length ($d$), with a stout elastic ring looped in at each end, to be adjusted, when wanted, to a pair of ranging poles, so as to give the horizontal distance; the poles themselves at the same time being held upright by means of *annular* plumb bobs suspended to just above the men's hands; such contrivances are as various as the circumstances of the ground.

The setting out a curve with the Theodolite is an example of the universal applicability of one grand principle of Trigonometry, viz. that any right-lined triangle may be inscribed in a circle; and this being done, each side of the triangle will be the chord of an arc double of that which measures the opposite angle; that is, double the sine of that angle measured in the circle; therefore the sides of the triangle are to each other as the sines of the opposite angles measured in the same circle, and consequently as the sines of the same angles measured in the circle whose radius is that of the tables. Hence the following proposition of such frequent use in the practice of Trigonometry.

In any triangle, the sines of the angles measured by any one circle are proportional to the sides opposite those same angles.

In our plan of curving, every side of a triangle (say A 1 2) *inscribed in a circle* is seen in practice to be the chord of the arc upon which the opposite angle of the same triangle stands, or double the sine of half the angle on it at the centre: that is, each side of such triangle is double the sine of half its opposite angle, *measured* by the circumscribed circle; thus there is established a direct relation between the sides and opposite angles of *any* triangle, shewn in this instance by the relations between the radius, the subchords, and the angle $\beta$.

Again, in one of our preliminary steps, suppose a radius is fixed upon, and we have to determine its proper tangent points, then

In the triangle AOC,

*Given* 1. Direction of the tangents, thence *the angles C, O**

2. *The side AO = r*

*To find AC = BC. Hence the points A, B*

Here are two angles and a side; and since the proportional sines of these angles given in the tables (or the actual sines either, only they are not so available) have the same relation to each other as the side given to the side required. We find the latter by the Golden Rule.

First say C is found = 47°
Then O must be = 43°
Given radius = 30 chains, then

| | | |
|---|---|---|
| As log. sin. ACO 47° - - - | = | 9.8641275 |
| : — — AOC 43° - - - | = | 1.4771213 |
| : : log. AO 30 chains - - - | = | 9.8337833 |
| | | 11.3109046 |
| : — AC or BC, 27.9765, or 27 chs. 97⅔ links from C - - - | = | 1.4467717 |

In order to become familiar with this cardinal principle, we recommend the student to inscribe triangles of various proportional sides in circles, joining the centre of the latter to the angular points of the former.

* (Halves of the great angles of C, O supplement to each other.)